Section 5.3
Trigonometric Functions of Any Angle

Could you repeat that?

There are many repetitive patterns in nature. Tides cycle through a pattern of low and high tides in a very predictable manner. The number of hours of daylight on a given day varies throughout the year, but the pattern throughout the year repeats itself year after year.

In this section, we extend the definitions of the trigonometric functions to include all angles. In doing so, we begin to see the repetitive properties of the trigonometric functions that make them useful for modeling cyclic phenomena.

Objective #1: Use the definitions of the trigonometric functions of any angle.

✔ *Solved Problem #1*

1a. Let $P = (1, -3)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

We are given that $x = 1$ and $y = -3$. We need the value of r.

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$$

Now we use the definitions of the trigonometric functions of any angle.

$$\sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{10}} = \frac{-3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{1} = -3$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{10}}{-3} = -\frac{\sqrt{10}}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{10}}{1} = \sqrt{10}$$

$$\cot\theta = \frac{x}{y} = \frac{1}{-3} = -\frac{1}{3}$$

✎ *Pencil Problem #1* ✐

1a. Let $P = (-2, -5)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

1b. Evaluate, if possible: $\csc 180°$.

The terminal side of $\theta = 180°$ is on the negative x-axis. We select the point $(-1, 0)$ on the terminal side of the angle, which is 1 unit from the origin, so $x = -1$, $y = 0$, and $r = 1$.

$$\csc\theta = \frac{r}{y} = \xcancel{\frac{1}{0}}$$

$\csc 180°$ is undefined.

1b. Evaluate, if possible: $\tan\frac{3\pi}{2}$.

Objective #2: Use the signs of the trigonometric functions.

✔ *Solved Problem #2*

2a. If $\sin\theta < 0$ and $\cos\theta < 0$, name the quadrant in which θ lies.

When $\sin\theta < 0$, θ lies in quadrant III or IV. When $\cos\theta < 0$, θ lies in quadrant II or III. When both conditions are met, θ must lie in quadrant III.

2b. Given that $\tan\theta = -\frac{1}{3}$ and $\cos\theta < 0$, find $\sin\theta$ and $\sec\theta$.

Because both the tangent and cosine are negative, θ lies in quadrant II, where x is negative and y is positive.

$$\tan\theta = -\frac{1}{3} = \frac{y}{x} = \frac{1}{-3}$$

So, $x = -3$ and $y = 1$. Find r.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$$

Now use the definitions of the trigonometric functions of any angle.

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{10}}{-3} = -\frac{\sqrt{10}}{3}$$

✎ *Pencil Problem #2* ✎

2a. If $\tan\theta < 0$ and $\cos\theta < 0$, name the quadrant in which θ lies.

2b. Given that $\tan\theta = -\frac{2}{3}$ and $\sin\theta > 0$, find $\cos\theta$ and $\csc\theta$.

Objective #3: Find reference angles.

✔ ***Solved Problem #3***	✎ ***Pencil Problem #3*** ✎
3a. Find the reference angle for $\theta = 210°$. The angle lies in quadrant III. The reference angle is $\theta' = 210° - 180° = 30°$.	**3a.** Find the reference angle for $\theta = 160°$.
3b. Find the reference angle for $\theta = \frac{7\pi}{4}$. The angle lies in quadrant IV. The reference angle is $\theta' = 2\pi - \frac{7\pi}{4} = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}$.	**3b.** Find the reference angle for $\theta = \frac{5\pi}{6}$.
3c. Find the reference angle for $\theta = -240°$. The angle lies in quadrant II. The positive acute angle formed by the terminal side of θ and the x-axis is 60°. The reference angle is $\theta' = 60°$.	**3c.** Find the reference angle for $\theta = -335°$.
3d. Find the reference angle for $\theta = 665°$. Subtract 360° to find a positive coterminal angle less than 360°: 665° – 360° = 305°. The angle $\alpha = 305°$ lies in quadrant IV. The reference angle is $\alpha' = 360° - 305° = 55°$.	**3d.** Find the reference angle for $\theta = 565°$.

3e. Find the reference angle for $\theta = -\frac{11\pi}{3}$.

Add 4π to find a positive coterminal angle less than 2π: $-\frac{11\pi}{3} + 4\pi = -\frac{11\pi}{3} + \frac{12\pi}{3} = \frac{\pi}{3}$.

The angle $\alpha = \frac{\pi}{3}$ lies in quadrant I. The reference angle is $\alpha' = \frac{\pi}{3}$.

3e. Find the reference angle for $\theta = -\frac{11\pi}{4}$.

Objective #4: Use reference angles to evaluate trigonometric functions.

✔ *Solved Problem #4*

4a. Use a reference angle to find the exact value of $\sin 300°$.

A 300° angle lies in quadrant IV, where the sine function is negative. The reference angle is $\theta' = 360° - 300° = 60°$.

$$\sin 300° = -\sin 60° = -\frac{\sqrt{3}}{2}$$

4b. Use a reference angle to find the exact value of $\tan\frac{5\pi}{4}$.

A $\frac{5\pi}{4}$ angle lies in quadrant III, where the tangent function is positive. The reference angle is $\theta' = \frac{5\pi}{4} - \pi = \frac{5\pi}{4} - \frac{4\pi}{4} = \frac{\pi}{4}$.

$$\tan\frac{5\pi}{4} = +\tan\frac{\pi}{4} = 1$$

✎ *Pencil Problem #4* ✐

4a. Use a reference angle to find the exact value of $\cos 225°$.

4b. Use a reference angle to find the exact value of $\sin\frac{2\pi}{3}$.

4c. Use a reference angle to find the exact value of $\sec\left(-\frac{\pi}{6}\right)$.

A $-\frac{\pi}{6}$ angle lies in quadrant IV, where the secant function is positive. Furthermore, a $-\frac{\pi}{6}$ angle forms an acute of $\frac{\pi}{6}$ with the *x*-axis. The reference angle is $\theta' = \frac{\pi}{6}$.

$$\sec\left(-\frac{\pi}{6}\right) = +\sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$$

4c. Use a reference angle to find the exact value of $\tan\left(-\frac{\pi}{4}\right)$.

4d. Use a reference angle to find the exact value of $\cos\frac{17\pi}{6}$.

Subtract 2π to find a positive coterminal angle less than 2π: $\frac{17\pi}{6} - 2\pi = \frac{17\pi}{6} - \frac{12\pi}{6} = \frac{5\pi}{6}$.

A $\frac{5\pi}{6}$ angle lies in quadrant II, where the cosine function is negative. The reference angle is $\theta' = \pi - \frac{5\pi}{6} = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}$.

$$\cos\frac{17\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

4d. Use a reference angle to find the exact value of $\cot\frac{19\pi}{6}$.